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A possible explanation of the clash for black hole entropy in the extremal limit

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Abstract

It is shown that the classical entropy of the extremal black hole depends on two different limits procedures. If we first take the extremal limit and then the boundary limit, the entropy is zero; if we do it the other way round, we get the Bekenstein-Hawking entropy. By means of the brick wall model, the quantum entropy of scalar field in the extremal black hole background has been calculated for the above two different limits procedures. A possible explanation which considers the quantum effect for the clash of black hole entropy in the extremal limit is given.

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There has been a lot of papers discussing the entropy of the black hole in the extremal limit recently, but the results are different[1-8]. Attention has been focused on whether the formula of the Bekenstein-Hawking entropy is still valid or the entropy is zero for the extremal black hole (EBH). Since this clash affects not only the statistical interpretation of entropy but also the phase transition[9-12] of the black hole in the extremal limit, it is of interest to further discuss this problem and investigate why contradictory results have been obtained.

To illustrate the reason behind the contradiction, we study the two dimensional (2D) charged dilaton black hole(CDBH)[7,13-15]. The action is

$$I = -\int_M \sqrt{g} e^{-2\phi}[R + 4(\bigtriangledown\phi)^2 + \lambda^2 - \frac{1}{2}F^2] - 2\int_{\partial M} e^{-2\phi} K \tag{1}$$

has a black hole solution metric

$$\mathrm{d}s^2 = -g(r)\mathrm{d}t^2 + g^{-1}(r)\mathrm{d}r^2 \tag{2}$$

$$g(r) = 1 - 2me^{-\lambda r} + q^2 e^{-2\lambda r} \tag{3}$$

$$e^{-2\phi} = e^{-2\phi_0} e^{\lambda r},\ A_0 = \sqrt{2} q e^{-\lambda r} \tag{4}$$

where m and q are the mass and electric charge of the black hole respectively. The horizons are located at $r_\pm = (1/\lambda)\ln(m \pm \sqrt{m^2 - q^2})$.

Using the finite-space formulation of black hole thermodynamics, employing the grand canonical emsemble and putting the black hole into a cavity as usual[3,7,16], we calculate the free energy and entropy of the CDBH. To simplify our calculations, we introduce a coordinate transformation

$$r = \frac{1}{\lambda}\ln[m + \frac{1}{2}e^{\lambda(\rho+\rho_0^*)} + \frac{m^2 - q^2}{2} e^{-\lambda(\rho+\rho_0^*)}] \tag{5}$$

where ρ_0^* is an integral constant, and rewrite Eq.(2) to a particular gauge

$$\mathrm{d}s^2 = -g_{00}(\rho)\mathrm{d}t^2 + \mathrm{d}\rho^2 \tag{6}$$

Transformation in Eq.(5) can be used for both the non-extremal black hole(NEBH) and the EBH, because in the extremal limit, the horizon ρ_+ and ρ_- become degenerate and are equal to $\rho_+ = (1/\lambda)\ln\sqrt{m^2 - q^2} - \rho_0^*$, so it changes to negative infinity and satisfies the topological requirement of the EBH. The Euclidean action takes the form

$$I = -\int_{\partial M} \sqrt{\frac{1}{g_{11}}} e^{-2\phi}(\frac{1}{2}\frac{\partial_1 g_{00}}{g_{00}} - 2\partial_1\phi) \tag{7}$$

The dilaton charge is found to be

$$D = e^{-2\phi_0}(m + \frac{1}{2}e^x + \frac{m^2 - q^2}{2}e^{-x}) \tag{8}$$

$$x = \lambda(\rho + \rho_0^*) \tag{9}$$

The free energy, $F = I/\beta$, where β is the proper periodicity of Euclideanized time at a fixed value of the special coordinate and has the form $\beta = 1/T_w = \sqrt{g_{00}}/T_c$ is the inverse periodicity of the Euclidean time at the horizon

$$T_c = \frac{\lambda\sqrt{m^2 - q^2}}{2\pi(m + \sqrt{m^2 - q^2})} \tag{10}$$

Using the formula of entropy $S = -(\partial F/\partial T_w)_D$, we obtain

$$S = \frac{2\pi e^{-2\phi}[m + \frac{e^x}{2} + \frac{(m^2-q^2)e^{-x}}{2}][1 + (m^2 - q^2)e^{-2x}]\sqrt{m^2-q^2}(m + \sqrt{m^2-q^2})}{(m^2 - q^2) + m[\frac{e^x}{2} + \frac{(m^2-q^2)e^{-x}}{2}]} \tag{11}$$

Taking the boundary limit $x \to x_+ = \lambda(\rho_+ + \rho_0^*) = \ln\sqrt{m^2 - q^2}$ in Eq.(11) to get the entropy of the hole, we find

$$S = 4\pi e^{-2\phi_0}(m + \sqrt{m^2 - q^2}) \tag{12}$$

This is just the result given by Nappi and Pasquinucci[14] for the non-extremal CDBH, which confirms that our treatment above is right for the entropy of the black hole.

We are now in a position to entend the above calculations to EBH. We are facing two limits, namely, the boundary limit $x \to x_+$ and the extremal limit $q \to m$. we can take the limits in different orders: (A) by first taking the boundary limit $x \to x_+$, and then the extremal limit $q \to m$; and (B) by first taking the extremal limit $q \to m$ and then the boundary limit $x \to x_+$. Obviously, taking the limits in these two different orders corresponds to two different treatments. In case (A), we first put a NEBH in a cavity and calculate its entropy, and then by taking the extremal condition to make the NEBH become EBH. In this treatment, we inherit the non-extremal topology of the black hole until to take the extremal limit. In case (B), we put an EBH into a cavity and initiate our work with the extremal topology. This treatment is similar to that of refs.[1,2]. To do our limits procedures mathematically, we may take $x = x_+ + \epsilon, \epsilon \to 0^+$ and $m = q + \eta, \eta \to 0^+$, where ϵ and η are infinitesimal quantities with different orders of magnitude, and substitute them

into Eq.(11). It can easily be shown that in case (A)

$$S_{cl}(A) = 4\pi m e^{-2\phi_0} \tag{13}$$

which is just the Bekenstein-Hawking entropy. However, in case (B),

$$S_{cl}(B) = 0 \tag{14}$$

which is just the result given by refs.[1,2]. Therefore, we have come to a conclusion that the two different results in fact come from taking limits in diffreent orders. Our results are in consistant with that given by Ghosh and Mitra recently[17].

Unfortunately, we then get an entropy clash. In statistical physics and thermodynamics, entropy is a function of an equilibrium state only, and does not depend on the history or the process how the system arrives at the equilibrium state as well as the different treatment of mathematics. From our discussions for the same EBH final state above, for different case (A) and (B) by taking limits in different orders, we find two different values for the entropy, namely the Bekenstein-Hawking entropy and zero. Of course, it would be most important to solve this clash.

Notice that the above discussions are limited in the classical relativity. We recall that the Gibbs paradox of the entropy for a mixing ideal gas which had also appeared in classical statistics and had been solved in quantum physics by considering the effects of identical particle. We hope to discuss the quantum entropy of the black hole and to see whether the quantum consideration will give some understanding of this clash.

An early suggestion by 't Hooft[18] was that the fields propagating in the region just outside the horizon give the main contribution to the black hole entropy. The entropy arises from entanglement[19,20]. Many methods, for example, the brick wall model[18,5], Pauli-Villars regulator theory[6],etc., have been suggested to calculate the quantum effects of entropy in WKB approximation or in the one-loop approximation. Suppose the CDBH is enveloped by a scalar field, and the whole system, the hole and the scalar field, are filling in a cavity. The wave equation of the scalar field is

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - M^2\phi = 0 \tag{15}$$

Substituting the metric Eq.(2) into Eq.(15), we find

$$E^2(1-2me^{-\lambda r}+q^2e^{-2\lambda r})^{-1}f + \frac{\partial}{\partial r}[(1-2me^{-\lambda r}+q^2e^{-2\lambda r})\frac{\partial f}{\partial r}] - M^2 f = 0 \tag{16}$$

Introducing the brick wall boundary condition[18]

$$\phi(x) = 0 \text{ at } r = r_+ + \epsilon$$

$$\phi(x) = 0 \text{ at } r = L$$

and calculating the wave number $K(r, E)$ and the free energy F, we get

$$K^2(r, E) = (1 - 2me^{-\lambda r} + q^2 e^{-2\lambda r})^{-1}[(1 - 2me^{-\lambda r} + q^2 e^{-2\lambda r})^{-1}E^2 - M^2] \tag{17}$$

$$F_{QM} = \frac{\pi}{6\beta^2\lambda}[\frac{1}{2}\ln(R^2 - 2mR + q^2) + \frac{m}{2\sqrt{m^2 - q^2}} \ln \frac{R - m - \sqrt{m^2 - q^2}}{R - m + \sqrt{m^2 - q^2}}] \tag{18}$$

where $R = e^{\lambda(r_+ + \epsilon)}$, and $\epsilon \to 0$ is the coordinate cutoff parameter. To extend the above discussion to EBH, we are facing two limits $\epsilon \to 0$ and $q \to m$ again. It can be proved that Eq(18) depends on the order of taking these two limits. We find for case (A) which we take $\epsilon \to 0$ (i.e. $r \to r_+$) at first and the extremal limit afterwards,

$$F_{QM}(A) = -\frac{\pi}{6\beta^2\lambda}\ln(\frac{1}{m\lambda\epsilon}) \tag{19}$$

but for case (B) which we adopt the extremal condition first and take $\epsilon \to 0$ afterwards

$$F_{QM}(B) = -\frac{\pi}{6\beta^2\lambda}(\frac{m}{m\lambda\epsilon} + \ln\frac{1}{m\lambda\epsilon}) \tag{20}$$

Similar to the classical case, different expressions for free energy appear here due to different priority of taking different limits. In order to compare with the other results, we replace the coordinate variable r by a proper variable ρ through $\mathrm{d}\rho = \mathrm{d}r/\sqrt{g(r)}$[18], we find that the proper cutoff ϵ' satisfies

$$\epsilon = \frac{\sqrt{m^2 - q^2}\lambda\epsilon'^2}{2(m + \sqrt{m^2 - q^2})} \tag{21}$$

The linear divergence becomes quadratic. Introducing

$$\tilde{\epsilon} = \frac{\sqrt{m^2 - q^2}\lambda\epsilon'^2}{2} \tag{22}$$

and substituting Eq.(22) into Eqs.(21),(20) and (19), we find Eqs(19)(20) can be reexpressed in the forms

$$F_{QM}(A) = -\frac{\pi}{6\beta^2\lambda}\ln\frac{1}{\lambda\tilde{\epsilon}} \tag{23}$$

$$F_{QM}(B) = -\frac{\pi}{6\beta^2\lambda}(\frac{m}{\tilde{\epsilon}} + \ln\frac{1}{\lambda\tilde{\epsilon}}) \tag{24}$$

It is easy to find that the coordinate transformations do not change the different divergent behavior appeared in Eqs(19)(20) got from case (A) and case (B). Here $\tilde{\epsilon}$ is also an infinitesimal quantity or $\tilde{\epsilon} \to 0$. The basic difference is that $(\tilde{\epsilon})^{-1}$ includes not only the ultraviolet divergence from the boundary condition but also the divergence from the extremal limit. Through the entropy formula $S = \beta^2(\partial F/\partial\beta)$, we obtain

$$S_{QM}(A) = \frac{\pi}{3\beta\lambda} \ln \frac{1}{\lambda\tilde{\epsilon}} \tag{25}$$

$$S_{QM}(B) = \frac{\pi}{3\beta\lambda}(\frac{m}{\tilde{\epsilon}} + \ln \frac{1}{\lambda\tilde{\epsilon}}) \tag{26}$$

Comparing Eqs.(13)(14) with Eqs.(25)(26), we find that instead of a term being proportional to the mass m in the case (A) for the classical black hole [Eq.(13)], we also find an additional term which is proportional to m but linearly divergent for $\tilde{\epsilon}$ in the case (B) with quantum correction for the black hole [Eq.(26)]. we adopt the viewpoint of the entanglement entropy [6,19,20] that the thermodynamical entropy of the black hole system is in fact the sum $S = S_{cl} + S_{QM}$ so we finally obtain

$$S(A) = S_{cl}(A) + S_{QM}(A) = 4\pi e^{-2\phi_0} m + \frac{\pi}{3\beta\lambda} \ln \frac{1}{\lambda\tilde{\epsilon}} \tag{27}$$

$$S(B) = S_{cl}(B) + S_{QM}(B) = \frac{\pi m}{3\beta\lambda\tilde{\epsilon}} + \frac{\pi}{3\beta\lambda} \ln \frac{1}{\lambda\tilde{\epsilon}} \tag{28}$$

Besides the same logarithmically divergent terms of $\tilde{\epsilon}$, the other two terms in Eqs(27) and (28) are both proportional to the mass of the black hole. This result is obviously in consistent with the results of refs.[5,22] in which they have argued that $S = km$ from thermodynamics and statistical physics for EBH and k is an undetermined constant. However, as shown in Eq.(28), the additional term of the entropy of scalar field includes the divergent factor $(1/\tilde{\epsilon})$.

As was pointed out earlier by Susskind and Uglum[23], the divergence of the quantum entropy can be removed by renormalization of gravitational coupling $\tilde{G}$ for the Schwarzchild black hole. Even though many authors have extended the argument of $\tilde{G}$ renormalization to other NEBH[6,19], the renormalization for EBH has not yet been realized. Note that the first term of the right hand side of Eq.(28) has two factors, namely, $1/\beta$ and $\tilde{\epsilon}$, and the temperature $1/\beta$ of EBH is zero. If one can prove that $1/\beta$ and $\tilde{\epsilon}$ have the same rate to reach zero after some "renormalizations", this term will be finite and may compensate the term $4\pi e^{-2\phi_0} m$ of the classical EBH because, in fact, the proportionality coefficient k of the entropy and the mass cannot be well determined for the EBH[5]. Then, we get a

possible explanation of the clash of the classical black hole in the extremal limit. After taking the quantum effects into account, the entropy is a function of the equilibrium state and is independent of the limits procedures as well as the processes through which the final state is reached. A complete classical treatment of the entropy of the black hole seems to be insufficient, and the quantum entropy must be involved. If we can find an expression of the entropy of the whole black hole system through a suitable parameter which not only depends on ϵ and $q \to m$, but also on $\bar{h}$, we speculate perhaps this clash can be resolved. Of course this is very difficult because we have not yet got a successful quantum gravitational theory.

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